Vertex-transitive CIS graphs

Ademir Hujdurović (University of Primorska, Slovenia)

Joint work with Edward Dobson, Martin Milanič and Gabriel Verret.

16.06.2015.

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

Naložba v vašo prihodnost
OPERACIJO DELNO FINANCIRA EVROPSKA UNIJA
Evropski socialni sklad

Overview

- CIS graphs
- Characterization of vertex-transitive CIS graphs
- Classification of vertex-transitive CIS graphs with maximal cliques of size 2 or 3
- Classification of vertex-transitive CIS graphs of valency at most 7

CIS graphs

Let Γ denote a finite simple undirected graph.
A **clique** (respectively, a **stable set**) of Γ is a set of pairwise adjacent (respectively, non-adjacent) vertices.

CIS graphs

Let Γ denote a finite simple undirected graph.
A **clique** (respectively, a **stable set**) of Γ is a set of pairwise adjacent (respectively, non-adjacent) vertices.
The inclusion maximal cliques and stable sets of Γ are called **maximal cliques** and **maximal stable sets** respectively.
The maximal cardinality of a clique (respectively a stable set) of Γ is called the **clique** (respectively **stability**) number and denoted $\omega(\Gamma)$ (respectively $\alpha(\Gamma)$).

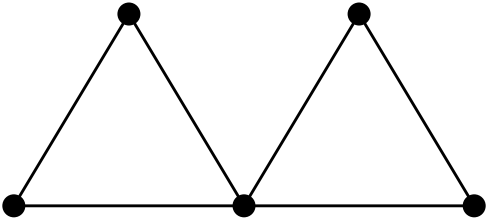
CIS graphs

Let Γ denote a finite simple undirected graph.
A **clique** (respectively, a **stable set**) of Γ is a set of pairwise adjacent (respectively, non-adjacent) vertices.
The inclusion maximal cliques and stable sets of Γ are called **maximal cliques** and **maximal stable sets** respectively.
The maximal cardinality of a clique (respectively a stable set) of Γ is called the **clique** (respectively **stability**) number and denoted $\omega(\Gamma)$ (respectively $\alpha(\Gamma)$).

Definition

A *CIS* graph is a graph in which every maximal stable set and every maximal clique intersect (CIS stands for "Cliques Intersect Stable sets").

Example



Basic properties of CIS graphs

- A graph is CIS if and only if its complement is CIS.

Basic properties of CIS graphs

- A graph is CIS if and only if its complement is CIS.
- A disconnected graph is CIS if and only if each of its connected component is CIS.

Basic properties of CIS graphs

- A graph is CIS if and only if its complement is CIS.
- A disconnected graph is CIS if and only if each of its connected component is CIS.
- For every two graphs Γ_1 and Γ_2, the lexicographic product $\Gamma_1[\Gamma_2]$ is CIS if and only if Γ_1 and Γ_2 are CIS.

The *lexicographic product* of graphs Γ_1 and Γ_2 is the graph $\Gamma_1[\Gamma_2]$ with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where two vertices (u, x) and (v, y) are adjacent if and only if either $\{u, v\} \in E(\Gamma_1)$ or $u = v$ and $\{x, y\} \in E(\Gamma_2)$.

(Co)-well-covered graphs

A graph Γ is *well-covered* if all its maximal stable sets are of the same size.
A graph Γ is *co-well-covered* if all its maximal cliques are of the same size. (Equivalently: if its complement is well-covered.)

Vertex-transitive graphs

A graph is called **vertex-transitive** if, for every pair of vertices, there exists an automorphism of the graph mapping one to the other.

A graph is called **vertex-transitive** if, for every pair of vertices, there exists an automorphism of the graph mapping one to the other.
Let G be a finite group with identity element 1, and let $S \subset G \backslash \{1\}$ be such that $S^{-1} = S$. We define the **Cayley graph** $Cay(G, S)$ on the group G with respect to the connection set S, to be the graph with vertex set G, and edges of the form $\{g, gs\}$ for $g \in G$ and $s \in S$.

A graph is called **vertex-transitive** if, for every pair of vertices, there exists an automorphism of the graph mapping one to the other.
Let G be a finite group with identity element 1, and let $S \subset G \backslash \{1\}$ be such that $S^{-1} = S$. We define the **Cayley graph** $Cay(G, S)$ on the group G with respect to the connection set S, to be the graph with vertex set G, and edges of the form $\{g, gs\}$ for $g \in G$ and $s \in S$.
A **circulant** is a Cayley graph on a cyclic group.

A graph is called **vertex-transitive** if, for every pair of vertices, there exists an automorphism of the graph mapping one to the other.
Let G be a finite group with identity element 1, and let $S \subset G \backslash \{1\}$ be such that $S^{-1} = S$. We define the **Cayley graph** $Cay(G, S)$ on the group G with respect to the connection set S, to be the graph with vertex set G, and edges of the form $\{g, gs\}$ for $g \in G$ and $s \in S$.
A **circulant** is a Cayley graph on a cyclic group.
Every Cayley graph is vertex-transitive. The converse is not true (smallest example is the Petersen graph).

Characterization of vertex-transitive CIS graphs

Theorem (Boros, Gurvich, Milanič, 2014)

Let Γ be a circulant graph. Then Γ is a CIS graph if and only if

1. *all maximal stable sets are of size $\alpha(\Gamma)$,*
2. *all maximal cliques are of size $\omega(\Gamma)$,*
3. $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$.

Characterization of vertex-transitive CIS graphs

Theorem (Boros, Gurvich, Milanič, 2014)

Let Γ be a circulant graph. Then Γ is a CIS graph if and only if

1. *all maximal stable sets are of size $\alpha(\Gamma)$,*
2. *all maximal cliques are of size $\omega(\Gamma)$,*
3. $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$.

Theorem (Dobson, H, Milanič, Verret, 2015)

Let Γ be a vertex-transitive graph. Then Γ is a CIS graph if and only if

1. *all maximal stable sets are of size $\alpha(\Gamma)$,*
2. *all maximal cliques are of size $\omega(\Gamma)$,*
3. $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$.

Examples of vertex-transitive CIS graphs

Proposition

If $n \geq 1$ then $L(K_{n,n})$ is a connected vertex-transitive CIS graph of order n^2 and valency $2(n-1)$ with $\alpha(L(K_{n,n})) = \omega(L(K_{n,n})) = n$.

Proposition

If $n \geq 1$ then $L(K_{n,n})$ is a connected vertex-transitive CIS graph of order n^2 and valency $2(n-1)$ with $\alpha(L(K_{n,n})) = \omega(L(K_{n,n})) = n$.

Proposition

Let $n \geq 2$ and let R_n be the Cayley graph on $\mathbb{Z}_{2n} \times \mathbb{Z}_4$ with connection set
$S = \{(0,1),(0,3),(n,0),(n,2),(2i,2),(2i+1,0) \mid 0 \leq i \leq n-1\}$.
Then R_n is a connected CIS graph of order $8n$ and valency $2n+3$ with $\alpha(R_n) = 2n$ and $\omega(R_n) = 4$.

Proposition

Let $n \geq 2$ and let S_n be the Cayley graph on $\mathbb{Z}_{2n} \times \mathbb{Z}_4$ with connection set
$S = \{(0,1),(0,3),(2i+1,0),(2i+1,1),(2i+1,3) \mid 0 \leq i \leq n-1\}$.
Then S_n is a connected CIS graph of order $8n$ and valency $3n+2$ with $\alpha(S_n) = 2n$ and $\omega(S_n) = 4$.

Examples of vertex-transitive CIS graphs

Proposition

Let $n \geq 2$ and let S_n be the Cayley graph on $\mathbb{Z}_{2n} \times \mathbb{Z}_4$ with connection set
$S = \{(0,1),(0,3),(2i+1,0),(2i+1,1),(2i+1,3) \mid 0 \leq i \leq n-1\}$.
Then S_n is a connected CIS graph of order $8n$ and valency $3n+2$ with $\alpha(S_n) = 2n$ and $\omega(S_n) = 4$.

All of the above examples are Cayley graphs. Does there exist a vertex-transitive CIS graph which is not a Cayley graph?

Vertex-transitive CIS graphs which are not Cayley

Definition

For $n \geq 3$, let $PX(n)$ be the graph with vertex-set $\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and edge-set $\{(i,x,y),(i+1,y,z) \mid i \in \mathbb{Z}_n,\ x,y,z \in \mathbb{Z}_2\}$. To $PX(n)$, we add the following set of edges $\{(i,x,y),(i,u,v) \mid i \in \mathbb{Z}_n,\ x,y,u,v \in \mathbb{Z}_2,\ (x,y) \neq (u,v)\}$ to obtain the graph Q_n.

Definition

For $n \geq 3$, let $PX(n)$ be the graph with vertex-set $\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and edge-set $\{(i,x,y),(i+1,y,z) \mid i \in \mathbb{Z}_n,\ x,y,z \in \mathbb{Z}_2\}$. To $PX(n)$, we add the following set of edges $\{(i,x,y),(i,u,v) \mid i \in \mathbb{Z}_n,\ x,y,u,v \in \mathbb{Z}_2,\ (x,y) \neq (u,v)\}$ to obtain the graph Q_n.

Proposition

Let $n \geq 4$. The graph Q_n is a connected vertex-transitive CIS graph of order $4n$ and valency 7 with $\alpha(Q_n) = n$ and $\omega(Q_n) = 4$. Moreover, if n is prime then Q_n is not a Cayley graph.

Using Gordon Royle's table of vertex-transitive graphs of order at most 32, we obtain the following with the help of a computer.

Proposition

Let $\mathcal{F}$ be the family containing the following graphs:

1. K_n, $n \geq 1$,
2. $L(K_{n,n})$, $n \geq 3$,
3. Q_n, $n \geq 4$,
4. R_n, $n \geq 3$,
5. S_n, $n \geq 2$,

and let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ under the operations of taking complements and lexicographic products. Then, up to isomorphism, every vertex-transitive CIS graph of order at most 32 is in $\overline{\mathcal{F}}$.

VT CIS graphs with $\omega = 2$ or $\omega = 3$

VT CIS graphs with $\omega = 2$ or $\omega = 3$

Theorem

Let Γ be a connected CIS graph with $\omega(\Gamma) = 2$. Then Γ is complete bipartite.

VT CIS graphs with $\omega = 2$ or $\omega = 3$

Theorem

Let Γ be a connected CIS graph with $\omega(\Gamma) = 2$. Then Γ is complete bipartite.

Corollary

Let Γ be a connected vertex-transitive CIS graph with $\omega(\Gamma) = 2$. Then $\Gamma \cong K_{n,n}$, for some positive integer n.

VT CIS graphs with $\omega = 2$ or $\omega = 3$

Theorem

Let Γ be a connected CIS graph with $\omega(\Gamma) = 2$. Then Γ is complete bipartite.

Corollary

Let Γ be a connected vertex-transitive CIS graph with $\omega(\Gamma) = 2$. Then $\Gamma \cong K_{n,n}$, for some positive integer n.

Theorem

Let Γ be a connected vertex-transitive graph with $\omega(\Gamma) = 3$. Then Γ is a CIS graph if and only if

1. $\Gamma \cong K_3[nK_1]$, *or*
2. $\Gamma \cong L(K_{3,3})[nK_1]$

for some positive integer n.

Theorem (Kostochka, 1980)

Let Γ be a graph and let $\mathcal{Q}$ be a collection of maximum cliques of Γ. Let $\Gamma_{\mathcal{Q}}$ be the graph of maximal cliques of Γ, that is, the graph with maximum cliques of Γ as vertices, and two such cliques are adjacent in $\Gamma_{\mathcal{Q}}$ if they intersect in Γ.
If $\omega(\Gamma) > \frac{2}{3}(\Delta(\Gamma)+1)$ and if $\Gamma_{\mathcal{Q}}$ is connected then $\cap\mathcal{Q} \neq \emptyset$.

Theorem (Kostochka, 1980)

Let Γ be a graph and let $\mathcal{Q}$ be a collection of maximum cliques of Γ. Let $\Gamma_{\mathcal{Q}}$ be the graph of maximal cliques of Γ, that is, the graph with maximum cliques of Γ as vertices, and two such cliques are adjacent in $\Gamma_{\mathcal{Q}}$ if they intersect in Γ.
If $\omega(\Gamma) > \frac{2}{3}(\Delta(\Gamma)+1)$ and if $\Gamma_{\mathcal{Q}}$ is connected then $\cap\mathcal{Q} \neq \emptyset$.

Lemma

Let Γ be a connected, k-regular, well-covered, co-well-covered graph. Then either $\omega(\Gamma) \leq \frac{2}{3}(k+1)$ or Γ is a complete graph.

Classification of VT CIS graphs with small valency

Theorem (Kostochka, 1980)

Let Γ be a graph and let $\mathcal{Q}$ be a collection of maximum cliques of Γ. Let $\Gamma_{\mathcal{Q}}$ be the graph of maximal cliques of Γ, that is, the graph with maximum cliques of Γ as vertices, and two such cliques are adjacent in $\Gamma_{\mathcal{Q}}$ if they intersect in Γ.
If $\omega(\Gamma) > \frac{2}{3}(\Delta(\Gamma)+1)$ and if $\Gamma_{\mathcal{Q}}$ is connected then $\cap\mathcal{Q} \neq \emptyset$.

Lemma

Let Γ be a connected, k-regular, well-covered, co-well-covered graph. Then either $\omega(\Gamma) \leq \frac{2}{3}(k+1)$ or Γ is a complete graph.

In particular, if Γ is not complete graph then:
if $k = 3$ then $\omega(\Gamma) \leq 2$
if $k = 4$ then $\omega(\Gamma) \leq 3$
if $k = 5$ or 6 then $\omega(\Gamma) \leq 4$
if $k = 7$ then $\omega(\Gamma) \leq 5$.

Theorem

Let Γ be a connected vertex-transitive graph of valency $k \leq 7$. Then Γ is a CIS graph if and only if Γ is isomorphic to one of the following graphs:

- $(k = 1)$ K_2;
- $(k = 2)$ K_3, $K_{2,2}$
- $(k = 3)$ K_4, $K_{3,3}$
- $(k = 4)$ K_5, $K_{4,4}$, $L(K_{3,3})$, $K_3[2K_1]$
- $(k = 5)$ K_6, $K_{5,5}$, $C_4[K_2]$,
- $(k = 6)$ K_7, $K_{6,6}$, $L(K_{4,4})$, $K_3[3K_1]$, $K_4[2K_1]$
- $(k = 7)$ K_8, $K_{7,7}$, $K_{3,3}[K_2]$, Q_n for $n \geq 4$.

Outline of the proof

If $k \leq 4$ the result follows from the result for $\omega = 2$ and $\omega = 3$.

Outline of the proof

If $k \leq 4$ the result follows from the result for $\omega = 2$ and $\omega = 3$.
If $k = 5$ and $\omega(\Gamma) = 4$, consider first the case when Γ is irreducible. The local graph of Γ has 5 vertices, maximal cliques of size 3 with no two triangles having an edge in common. There is only one such graph.

Outline of the proof

If $k \leq 4$ the result follows from the result for $\omega = 2$ and $\omega = 3$.
If $k = 5$ and $\omega(\Gamma) = 4$, consider first the case when Γ is irreducible. The local graph of Γ has 5 vertices, maximal cliques of size 3 with no two triangles having an edge in common. There is only one such graph.

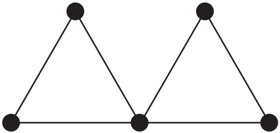


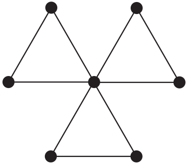
Figure: The graph T_2

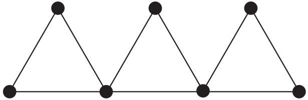
It is now easy to conclude that Γ is the lexicographic product of a cycle C_n ($n \geq 4$) with K_2. Then $n = 4$ and $\Gamma = C_4[K_2]$.

Outline of the proof

The cases when $(k, \omega) \in \{(6,4),(7,5)\}$ are solved similarly.

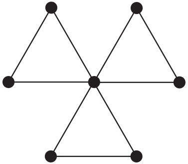
The cases when $(k, \omega) \in \{(6,4),(7,5)\}$ are solved similarly.
If $k = 7$ and $\omega(\Gamma) = 4$ then there are two possible local graphs:

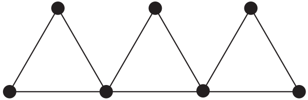




Outline of the proof

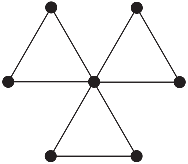
The cases when $(k, \omega) \in \{(6,4),(7,5)\}$ are solved similarly.
If $k = 7$ and $\omega(\Gamma) = 4$ then there are two possible local graphs:

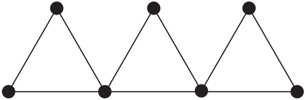




The first local graph is easy to handle, we have $\Gamma = K_{3,3}[K_2]$.

The cases when $(k, \omega) \in \{(6,4),(7,5)\}$ are solved similarly.
If $k = 7$ and $\omega(\Gamma) = 4$ then there are two possible local graphs:





The first local graph is easy to handle, we have $\Gamma = K_{3,3}[K_2]$.
The case of the second local graph is much more difficult, and it results in the infinite family Q_n.

Open problems

Question

Does every CIS graph Γ *satisfy* $\alpha(\Gamma)\omega(\Gamma) \geq |V(\Gamma)|$*?*

Open problems

Question

Does every CIS graph Γ *satisfy* $\alpha(\Gamma)\omega(\Gamma) \geq |V(\Gamma)|$*?*

Question

Does the chromatic number of every vertex transitive CIS graph equal its clique number?

Open problems

Question

Does every CIS graph Γ satisfy $\alpha(\Gamma)\omega(\Gamma) \geq |V(\Gamma)|$?

Question

Does the chromatic number of every vertex transitive CIS graph equal its clique number?

Question

Is every connected regular irreducible well-covered co-well-covered CIS graph vertex-transitive?

Thank you!!!